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# Practical Inversion Formulas

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# Practical Inversion Formulas

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**Abstract**

Various inversion formulas in terms of characteristic functions, moments and real Laplace transforms are studied from the viewpoint of practical applicability. A new inversion integral in terms of the characteristic function for integer-valued variables and a new moment inversion formula for variables in the unit interval are given.

## 1. Introduction and summary

Let $X$ be a random variable with distribution function $F(x)=P(X\leq x)$ and characteristic function $\varphi(t)=E(e^{itX})=r(t)\,e^{i\theta(t)}$, where $r(t)$ and $\theta(t)$ are real. It is not necessary that $r(t)$ is non-negative or that $\theta(t)$ belongs to some particular interval. Put $\mu_\beta=E(X^\beta)$ if this moment is well defined.

For the case that $X$ is integer-valued we present in section 2 a new inversion formula analogous with the Lévy inversion integral. In section 3 we make some remarks on Lévy's inversion formula from the viewpoint of numerical and theoretical applications to non-negative random variables. In section 4 Lévy's formula is applied to distributions concentrated to $[0, 1]$. This yields an inversion formula in terms of moments, since the characteristic function is determined by the moments in this case. For a non-negative random variable $Y$ this result immediately gives an inversion formula in terms of the real Laplace transform, by the transformation $X=e^{-Y}$. In section 5 this formula is analyzed and some numerical results presented. Feller (1971, p. 227) gave an inversion formula in terms of moments for distributions on $[0, 1]$. In section 6 we modify this formula to get a closer approximation to $F$ and we simplify the combinatorial expressions occurring in it.

The moment inversion formulas of sections 4 and 6 both give $F(x)$ as a limit as $A\to\infty$ of expressions of the type $c_A+c_{A,1}\mu_\beta+c_{A,2}\mu_{2\beta}+c_{A,3}\mu_{3\beta}+\ldots$, where $c_{A,r}$ alternate in sign with $r$ and $\max_r|c_{A,r}|\to\infty$ as $A\to\infty$. Consequently the limitation in applying these formulas lies in the computer's precision, i.e. the number of digits for the mantissa that can be stored in a data register representing a real number. We assume a decimal representation. The convergence properties as precision tends to infinity are studied. For the formula of section 4, the smoother $F$ is, the faster is the convergence. This is because $\varphi(t)$ tends to zero faster, the smoother $F$ is.

On the other hand, the formula of section 6 gives a slow convergence for continuous distributions, while for discrete distributions the rate of convergence is shown to be exponential if $F(x)$ is computed for $x$ between adjacent atoms of $F$. The formulas of sections 4 and 6 thus complement each other.

## 2. Inversion integrals for integer-valued variables

Assume that $X$ is integer-valued. Then we have the well-known formula

$$P(X=j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\,e^{-ijt}\,dt,\quad j=0,1,-1,2,-2,\ldots,$$

see e.g. Feller (1971, p. 511). Now the real part of the integrand is even and the imaginary part is odd, hence

$$P(X=j)=\frac{1}{\pi}\int_{0}^{\pi}r(t)\cos(\theta(t)-jt)\,dt.$$

By induction one can prove the identity

$$\sum_{j=m}^{n}e^{-ijt}=e^{-i\frac{1}{2}(m+n)t}\sin(\tfrac{1}{2}(n-m+1)\,t)/\sin(\tfrac{1}{2}t),\quad m\leqslant n.$$

This yields

$$\begin{aligned}P(m\leqslant X\leqslant n)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\,e^{-i\frac{1}{2}(m+n)t}\frac{\sin(\frac{1}{2}(n-m+1)\,t)}{\sin(\frac{1}{2}t)}\,dt\\&=\frac{1}{\pi}\int_{0}^{\pi}r(t)\cos(\theta(t)-\tfrac{1}{2}(m+n)\,t)\frac{\sin(\frac{1}{2}(n-m+1)\,t)}{\sin(\frac{1}{2}t)}\,dt,\end{aligned}$$

$$m\leqslant n,\ m\text{ and }n\text{ integers.}\quad (1)$$

For $t=0$ the integrand is to interpreted as $n-m+1$. The formula is for example suitable when $X$ is a linear combination of independent binomial variables.

If $F(x)=0$ for $x<0$ we can put $n=x$ and $m=-x$, which yields

$$F(x)=\frac{1}{\pi}\int_{0}^{\pi}\operatorname{Re}\varphi(t)\frac{\sin((x+\frac{1}{2})\,t)}{\sin(\frac{1}{2}t)}\,dt,\quad x=0,1,2,\ldots\quad (2)$$

This formula has an appealing simplicity, but for numerical purposes it would normally be better to use (1) with $n=x$ and $m=0$; the integrand in (2) oscillates more rapidly than in (1) and would thus need to be computed at more points for a satisfactory approximation by for example Simpson's formula.

**3. On Lévy's inversion integral**

The Lévy inversion formula can be written in the form

$$F(x)-F(y)=\lim_{A\to\infty}\frac{1}{\pi}\int_{-A}^{A}\varphi(t)\,e^{-i\frac{1}{2}(x+y)t}\sin(\tfrac{1}{2}(x-y)\,t)\,t^{-1}\,dt,\quad y<x,$$

if $F$ is continuous at $x$ and at $y$. See e.g. Cramér (1946, p. 93). The analogy with (1) is evident. As before the real and imaginary parts of the integrand are even and odd, respectively, hence

$$F(x)-F(y)=\frac{2}{\pi}\int_{0}^{\infty}r(t)\cos(\theta(t)-\tfrac{1}{2}(x+y)\,t)\sin(\tfrac{1}{2}(x-y)\,t)\,t^{-1}\,dt, \tag{3}$$

with the integral defined at least in the improper Riemann sense. If $F(x)=0$ for $x<0$ we can put $y=-x$, which yields

$$F(x)=\frac{2}{\pi}\int_{0}^{\infty}\mathrm{Re}\,\varphi(t)\sin(xt)\,t^{-1}\,dt,\quad x\geq 0, \tag{4}$$

if $x$ is a continuity point of $F$. This is analogous with (2). Putting $y=0$ (or $y=-1$ if $P(X=0)>0$) in (3) gives a more slowly oscillating integrand than in (4). The integrand in (4) might however tend to zero faster. For example, the exponential distribution $F(x)=1-e^{-x}$ has $\varphi(t)=(1-it)^{-1}$, $r(t)=(1+t^2)^{-1/2}$ and $\mathrm{Re}\,\varphi(t)=(1+t^2)^{-1}$. For theoretical purposes the simplicity of (4) is an advantage, and we shall use this representation as a basis for the moment inversion formula of the next section.

**4. Moment inversion for distributions on the unit interval**

Assume that $0\leq X\leq 1$. Let $\alpha$ be any positive number and let $\varphi_\alpha$ be the characteristic function of $X^\alpha$. Let

$$F_\alpha(y,A)=\frac{2}{\pi}\int_{0}^{A}\mathrm{Re}\,\varphi_\alpha(t)\,t^{-1}\sin(yt)\,dt,\quad y>0. \tag{5}$$

Then $F_\alpha(x^\alpha,A)\to F(x)$ as $A\to\infty$ if $F$ is continuous at $x$, by (4). Choosing $\alpha$ suitably might for fixed $A$ give a better approximation to $F(x)$ than just $\alpha=1$. We assume that the distribution of $X$ is known through the moments $\mu_\beta$, $\beta\geq 0$. We have for all $t$ the expansion

$$\mathrm{Re}\,\varphi_\alpha(t)=1+\sum_{r=1}^{\infty}(-1)^r\mu_{2\alpha r}\,t^{2r}/(2r)\,!. \tag{6}$$

It turns out that the required precision in computing (5) from (6) is approximately proportional to $A$, see (12). The smoother the distribution function of $X^\alpha$ is, the faster $\varphi_\alpha(t)$ tends to zero as $t$ tends to infinity and the faster the convergence to $F(x)$ is as $A$ tends to infinity. See Feller (1971, p. 514). As a rule of thumb one may choose $\alpha$ so that $\mu_\alpha=\frac{1}{2}$ (making $X^\alpha$ as nearly uniformly distributed as possible, roughly speaking), unless something else indicates otherwise. Let

$$g(z)=\frac{2}{\pi}\int_0^z t^{-1}\sin t\,dt \tag{7}$$

and

$$a_r(y,A)=\frac{2(-1)^r}{\pi(2r)!}\int_0^A t^{2r-1}\sin(yt)\,dt. \tag{8}$$

Then (5) can be written

$$F_\alpha(y,A)=g(yA)+\sum_{r=1}^{\infty}\mu_{2\alpha r}\,a_r(y,A). \tag{9}$$

We could replace $g(yA)$ with its limit 1 and still have $F_\alpha(x^\alpha, A)\to F(x)$, but the rate of convergence would in general be slower. The equation (9) holds even if $X$ is not bounded, provided (6) holds for some $t>A$.

We have the obvious bound

$$|a_r(y,A)|\leqslant\frac{A^{2r}}{\pi r(2r)!}. \tag{10}$$

This is actually a rather close estimate, unless $y$ is very small. For example, if the right side is $10^8$ the left side is normally at least $10^7$, and if the right side is $10^{32}$ the left side is normally at least $10^{30}$. The maximum of the right side of (10) is attained either for $r=[\frac{1}{2}A]$ or $r=[\frac{1}{2}A]-1$. Applying Stirling's formula we get

$$\max_r\frac{A^{2r}}{\pi r(2r)!}\sim 2^{1/2}(\pi A)^{-1.5}e^A \quad (A\to\infty). \tag{11}$$

Assume that the precision is $M$ decimal digits. Assume further that the computation entails a numerical error making the last $Q$ of the $M$ digits in a computed value for $\mu_{2\alpha r}a_r(y,A)$ unreliable. By (10) and (11) we can then compute $F_\alpha(y,A)$ with $T$ correct decimals if

$$M>T+Q+{}^{10}\log(2^{1/2}\pi^{-1.5})-1.5\,{}^{10}\log A+A\,{}^{10}\log e. \tag{12}$$

For $M=13$ and $T=Q=3$ this inequality is satisfied if $A\leqslant 22$. Note that $\mu_{2\alpha r}$ must be computed with sufficiently many correct decimals.

Solving the integral in (8) we obtain

$$a_r(y,A)=(\pi r y^{2r})^{-1}\sum_{k=0}^{r-1}(-1)^k\frac{(yA)^{2k}}{(2k)!}\left(\frac{yA\cos(yA)}{2k+1}-\sin(yA)\right). \tag{13}$$

Recognizing the first $r$ terms in the MacLaurin expansions of $\sin(yA)$ and $\cos(yA)$ we can also write

$$a_r(y,A)=-(\pi r y^{2r})^{-1}\sum_{k=r}^{\infty}(-1)^k\frac{(yA)^{2k}}{(2k)!}\left(\frac{yA\cos(yA)}{2k+1}-\sin(yA)\right). \tag{14}$$

From (14) we can deduce that

$$a_r(y,A)\sim\frac{(-1)^r\sin(yA)A^{2r}}{\pi r(2r)!}\quad(r\to\infty),$$

provided $\sin(yA)\neq 0$. It follows that $a_r(y,A)$ alternates in sign and decreases in absolute value for large $r$. The error in terminating the sum in (9) at $K$ will thus be smaller in absolute value than and have the same sign as the first omitted term $a_{K+1}(y,A)$, provided $K$ is large enough. Normally $K=[eA/2]+5$ will satisfy any desired degree of accuracy, by (10) and Stirling's formula. The maximum of $(yA)^{2k}/(2k)!$ is attained for $k=[(1+(4(yA)^2+1)^{1/2})/4]$. For maximal numerical accuracy we should use (13) for $r$ from 1 up to and including this value and (14) for the remaining values of $r$ up to $K$. The summation should go from smaller to larger values of $(yA)^{2k}/(2k)!$, i.e. forwards in (13) and backwards in (14). Using recursive schemes of computation in stages we can keep run times short. We can simplify $a_r(y,A)$ by putting $A=(\pi n+\frac{1}{2}\pi)/y$, $n=0,1,2,\ldots$ Then $\cos(yA)=0$ and $\sin(yA)=(-1)^n$. Furthermore $g(z)$ is close to 1 at the points $z=\pi n+\frac{1}{2}\pi$. For small $y$ we might not, however, be able to satisfy the inequality (12) with this choice of $A$. Formulas similar to (9) can be obtained from other variations of inversion integrals than (4). The ones tried by us have however been found to be less tractable numerically than (9).

## 5. Real Laplace inversion with numerical examples

If $Y$ is a non-negative random variable with distribution function $H$ and Laplace transform $\hat{H}(s)=E(e^{-sY})$, the transformation $X=e^{-Y}$ turns (9) into an inversion formula in terms of the real Laplace transform at a lattice of points. For $X$ we have $\mu_\beta=\hat{H}(\beta)$. For continuity points $y$ it holds $H(y)=1-F(e^{-y})$, and hence

$$H(y)=1-\lim_{A\to\infty}\left(g(e^{-\alpha y}A)+\sum_{r=1}^{\infty}\hat{H}(2\alpha r)\,a_r(e^{-\alpha y},A)\right). \tag{15}$$

Choose $\alpha$ so that $\hat{H}(\alpha)=\frac{1}{2}$, unless otherwise indicated (see below for such an indication).

For the implementation of this result we have developed a package of programs with optional precision up to several thousand digits for the personal computer ABC80. A program for Laplace inversion based on (13), (14) and (15) can handle $A \leqslant 195$. The summation in (15) is to $[e195/2]+5=270$. For $A=195$ we require the precision $M=89$ decimal digits in order to have $T+Q=8$, by (12). The restriction $A \leqslant 195$ is determined by internal memory capacity. Memory requirement increases with $A^2$. Another restriction is computer time. If the Laplace transform under inversion can be computed with a fixed number of evaluations of functions that are as numerically well-conditioned as elementary and cyclometric functions, then computing time increases with $A^4$. For such Laplace transforms, which constitute the bulk of applications, computing time for $A=195$ on the ABC80 is a matter of hours, so that in any case leaving the computer on overnight will do the job of computing $\hat{H}(2\alpha), \ldots, \hat{H}(2\alpha 270)$. Each value for the distribution function then takes about 10 minutes.

The programs can be obtained by request from the author. A variety of Laplace transforms have been inverted with good results. The worst cases are the tails of distribution functions $H$ with heavy tails. This corresponds to a large mass close to the origin for the transformed variable $X=e^{-Y}$. In the worst cases examined the error in the approximation of $H(y)$ was estimated to be about $5 \times 10^{-3}$. For these cases one should choose $\alpha$ so small that $e^{-\alpha y}A$ is not too small, preferably larger than $\pi$. The integrand in (5), with $y$ replaced by $e^{-\alpha y}$, needs to oscillate sufficiently over $(0, A)$. For nice cases, where $\varphi_\alpha(t) \to 0$ fast as $t \to \infty$, this oscillation is not needed. For distributions with light tails the error was smaller than $10^{-5}$. Examples of such nice distributions are:

I. The exponential distribution with $\hat{H}(s)=1/(1+s)$.

II. A linear combination of independent binomial variables (the distribution function has to be computed halfway between jump points).

III. The Laplace transform $\hat{H}(s)=4s^{1/2}/(e^{2s^{1/2}}-e^{-2s^{1/2}})$, which arises as a limit in critical branching processes. The following values were obtained among others.

| $y$ | 0.2 | 0.4 | 0.6 | 1.0 | 1.8 |
|---|---|---|---|---|---|
| $H(y)$ | 0.03400 | 0.29290 | 0.55028 | 0.83049 | 0.97644 |

For heavy-tailed distributions the rate of convergence in (15), although slower than for light-tailed ones, was found to be of the order $1/A$, anyway. Examples of heavy-tailed distributions are:

IV. The busy period distribution for the $G/M/2$ queue, with Laplace transform

$$\hat{H}(s) = \frac{\mu(1-\hat{F}(s+\mu))}{(s+\mu)(1-2\hat{F}(s+\mu))} - \frac{2\mu\hat{F}(s+\mu)(1-g(s))}{(s+2\mu-2\mu g(s))(1-2\hat{F}(s+\mu))},$$

where $\mu$ is the intensity of service, $\hat{F}(s)$ is the Laplace transform of the inter-arrival distribution and $g(s)$ is the solution in $(0, 1)$ of the equation $z=\hat{F}(s+2\mu-2\mu z)$. See Rosenlund (1978), eq. (16). This one belongs to that group of unwieldy transforms in queueing theory that are held to be nearly impossible to invert. We took $\hat{F}(s)=(\lambda/(s+\lambda))^{4.5}$ and the traffic intensity $\varrho=\lambda/9\mu=0.8$. Some obtained values, with $\alpha=0.4$ and $A=194$, follow.

| $\mu y$ | 0.1 | 0.25 | 0.75 | 1.25 | 2 | 5 | 10 |
|---|---|---|---|---|---|---|---|
| $H(y)$ | 0.095 | 0.219 | 0.447 | 0.547 | 0.637 | 0.799 | 0.883 |

The error should be smaller than $10^{-3}$ for $\mu y$ up to 2 and smaller than $5\times10^{-3}$ for $\mu y=5$ and $\mu y=10$.

V. The waiting time in the random order service $G/M/m$ queue, see Rosenlund (1980). The Laplace transform involves a numerically difficult integral, which, due to the time restriction, we could compute to only about 20 decimal digits, admitting $A=40$. Since the tail of this distribution is less heavy than in the preceding example, this gave an error of about $5\times10^{-3}$. Numerical results will be presented elsewhere.

## 6. On Feller's inversion for moments

Assume that $0\leqslant X\leqslant 1$ and let $F_\alpha(x)=F(x^{1/\alpha})$ be the distribution function of $X^\alpha$. Feller's (1971, p. 227) inversion formula applied to $F_\alpha$ can be formulated in the following way. Let $B_{n,\theta}$ be the distribution function for the binomial distribution with parameters $n$ and $\theta$, that is

$$B_{n,\theta}(x)=\sum_{j=0}^{[x]}\binom{n}{j}\theta^j(1-\theta)^{n-j},\quad x\geqslant 0. \tag{16}$$

Let

$$G_n^\alpha(t)=\int_0^1 B_{n,\theta}(nt)\,dF_\alpha(\theta),\quad 0\leqslant t\leqslant 1. \tag{17}$$

Since the limit as $n\to\infty$ of $B_{n,\theta}(nt)$ is 1 for $\theta<t$, $\frac{1}{2}$ for $\theta=t$ and 0 for $\theta>t$, we have

$$\lim_{n\to\infty} G_n^\alpha(x^\alpha)=\tfrac{1}{2}(F(x)+F(x-0)),\quad 0<x<1, \tag{18}$$

and for $x=0$ the limit is $F(0)$ and for $x=1$ it is 1. It seems to be best, in general, to choose $\alpha$ so that $\mu_\alpha=\frac{1}{2}$. Expanding $(1-\theta)^{n-j}$ in (16) according to the binomial theorem, putting (16) into (17) and interchanging integration and summation turns (18) into a moment inversion formula. Let

$$a_{n,k}^\alpha=\sum_{j=0}^{k}\binom{n}{j}\sum_{i=0}^{n-j}(-1)^i\binom{n-j}{i}\mu_{\alpha i+\alpha j}. \tag{19}$$

Then it holds

$$G_n^\alpha(t) = a^\alpha_{n,[nt]}. \tag{20}$$

We shall write (19) as a simple sum. Rearrangement gives ($\wedge$ for min)

$$a^\alpha_{n,k} = 1+\sum_{r=1}^{n}(-1)^r\binom{n}{r}\mu_{\alpha r}\sum_{j=0}^{k\wedge r}(-1)^j\binom{r}{j}.$$

Using the combinatorial identity

$$\sum_{j=0}^{k\wedge r}(-1)^j\binom{r}{j} = \begin{cases} 0, & 1\leqslant r\leqslant k \\ (-1)^k\binom{r-1}{k}, & 0\leqslant k<r \end{cases}$$

we get

$$a^\alpha_{n,k} = 1+\sum_{r=k+1}^{n}(-1)^{r+k}\binom{n}{r}\binom{r-1}{k}\mu_{\alpha r}, \quad 0\leqslant k<n. \tag{21}$$

Now $G_n^\alpha$ is a step function with jumps at $0, 1/n, 2/n, \ldots, 1$. The approximation to $F_\alpha(x)$ is improved if $G_n^\alpha$ is replaced by the continuous modification $H_n^\alpha$ obtained by putting $H_n^\alpha(0)=0$, $H_n^\alpha(1)=1$, $H_n^\alpha(t)=G_n^\alpha(t)$ when $t$ is the midpoint $k/n+1/2n$ in the interval $(k/n, (k+1)/n)$ where $G_n^\alpha$ is constant ($k=0, \ldots, n-1$), and $H_n^\alpha$ linear between these points. With

$$b^\alpha_{n,k}(x) = a^\alpha_{n,k}x+a^\alpha_{n,k-1}(1-x)$$

we have

$$b^\alpha_{n,k}(x) = 1+\sum_{r=k}^{n}(-1)^{r+k}\binom{n}{r}\binom{r}{k}(x-k/r)\mu_{\alpha r} \tag{22}$$

and the representation

$$H_n^\alpha(t) = \begin{cases} 2na^\alpha_{n,0}t & \text{for } 0\leqslant t\leqslant \dfrac{1}{2n} \\ b^\alpha_{n,[nt+\frac{1}{2}]}(nt+\tfrac{1}{2}-[nt+\tfrac{1}{2}]) & \text{for } \dfrac{1}{2n}\leqslant t\leqslant 1-\dfrac{1}{2n} \\ 1-2n\mu_{\alpha n}(1-t) & \text{for } 1-\dfrac{1}{2n}\leqslant t\leqslant 1. \end{cases} \tag{23}$$

It holds $H_n^\alpha(x^\alpha)\to F(x)$ at points of continuity. To illustrate the improvement achieved, consider the uniform distribution $F(x)=x$ and take $\alpha=1$. Then (16) and (17) give

$$G_n^1(t) = ([nt]+1)/(n+1),$$

$H_n^1(t) = (nt+\frac{1}{2})/(n+1), \frac{1}{2n} \leq t \leq 1-\frac{1}{2n}$,

which yields

$$\sup_{0\leq t\leq 1} |G_n^1(t)-t| = 1/(n+1),$$

$$\sup_{0\leq t\leq 1} |H_n^1(t)-t| = (n-1)/2n(n+1).$$

This example also shows that for continuous distributions $F$ we cannot in general hope for a better rate of convergence than $O(1/n)$. We shall, however, show that the rate of convergence $G_n^1(t)\to F(t)$ is exponential if $u<t<s$ and $P(u<X<s)=0$. We shall also show that the required precision is approximately proportional to $n$. For a discrete distribution $F$ the method of this section is thus at least asymptotically better (as precision tends to infinity) than that of the preceding section, provided the jump points are known and $F(u)$ is approximated by $H_n^\alpha(\frac{1}{2}(u^\alpha+s^\alpha))$, if $u<s$ are two adjacent jump points. From (17) we get

$$F(t)-G_n^1(t) = \int_0^t (1-B_{n,\theta}(nt))\, dF(\theta) - \int_t^1 B_{n,\theta}(nt)\, dF(\theta). \tag{24}$$

Now $B_{n,\theta}(nt)$ decreases with $\theta$. If $u<t<s$ we obtain

$$-(1-F(u))\, B_{n,s}(nt) \leq F(u)-G_n^1(t) \leq F(u)\,(1-B_{n,u}(nt)) \tag{25}$$

or equivalently

$$(G_n^1(t)-B_{n,s}(nt))/(1-B_{n,s}(nt)) \leq F(u) \leq G_n^1(t)/B_{n,u}(nt). \tag{26}$$

If $[x]\leq[\theta(n+1)]$ the terms of (16) increase with $j$. Since $t<s$ implies $[nt]\leq[s(n+1)]$ we obtain

$$B_{n,s}(nt) \leq ([nt]+1)\binom{n}{[nt]} s^{[nt]}(1-s)^{n-[nt]}$$
$$\sim \left(\frac{t(1-s)}{s(1-t)}\right)^{nt-[nt]} \left(\frac{nt}{2\pi(1-t)}\right)^{1/2} \left(\frac{s^t(1-s)^{1-t}}{t^t(1-t)^{1-t}}\right)^n \quad (n\to\infty). \tag{27}$$

The function $x^t(1-x)^{1-t}$ of $x$ has its maximum at $x=t$, so that the last factor is smaller than one. For the right side of (25) we observe that $1-B_{n,u}(nt)=B_{n,1-u}(n(1-t)-0)$. This proves the exponential rate of convergence in $G_n^1(t)\to F(t)$. The required precision in computing (22) is determined by the maximum of $\binom{n}{r}\binom{r}{k}$, which is attained for $r=[(2n+1)/3]$ and $k=[r/2]$. It is asymptotically $3^{n+1.5}/2\pi n$. Assume as in section 4 that the precision is $M$ decimal digits and that the computation entails a numerical error making the last $Q$ of the $M$ digits in a computed value for $\binom{n}{r}\binom{r}{k}(x{-}k/r)\mu_{\alpha r}$ unreliable. Taking into account also the factor $x{-}k/r$ we can then for all $t$ compute $H_n^\alpha(t)$ with $T$ correct decimals if

$$M > T + Q + {}^{10}\log(3^{1.5}/4\pi) - {}^{10}\log n + n\,{}^{10}\log 3. \qquad (28)$$

In computing (22) we note that the maximum of $\binom{n}{r}\binom{r}{k}$ for fixed $n$ and $k$ is attained for $r = [\frac{1}{2}(n+k+1)]$, which can be used to determine required precision when a single value of $H_n^\alpha(t)$ is wanted.

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Complement to section 5 in "Practical Inversion Formulas" by Stig I. Rosenlund

It is stated that for heavy-tailed distributions one should take $\alpha$ so small that $A\exp(-\alpha y) \geq \pi$. In numerical inversions it was found that $9.5\pi$ gave the best results; choosing $\alpha$ smaller or larger yielded larger deviations from known values for the distribution function. We thus recommend taking

$$\alpha = \min\left\{\hat{H}^{-1}(\tfrac{1}{2}), \log(A/9.5\pi)/y\right\}$$

for heavy-tailed distributions. For light-tailed ones it is best to let $\hat{H}(\alpha) = \frac{1}{2}$. If it is not known whether H is light-tailed or heavy-tailed, compute a few inverted values for some large y-values with $\alpha$ dependent on y as above. If H turns out to be light-tailed, recompute with $\alpha$ so that $\hat{H}(\alpha) = \frac{1}{2}$.

Experiments on some known continuous distributions, including one with $1 - H(y) \sim c/\sqrt{y}$, gave good results for arbitrarily large y with errors smaller than $10^{-3}$ for A = 195 and $\alpha$ chosen as above. Even heavy-tailed discrete distributions could be inverted with errors smaller than $10^{-2}$ for $y = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$ .

For example IV we obtained 0.798 and 0.894 for $\mu y$ equal to 5 and 10 with $\alpha$ as above. If 0.894 is better than the table's 0.883, then the latter is in error by more than $5 \times 10^{-3}$.

For continuous light-tailed distributions the error is much smaller, as remarked. In example III the precision of the values for H(y) was determined by comparison of inverted values for a sequence of A up to 195. Since then we have by theoretical inversion derived the expression

$$H(y) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp(-y(\tfrac{1}{2}n\pi)^2).$$

This formula gives the same values as in the table of example III.